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On certain morphisms of sequential dynamical systems

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Abstract

We study a class of discrete dynamical systems that consist of the following data: (a) a finite (labeled) graph Y with vertex set $\{1, \ldots, n\}$, where each vertex has a binary state, (b) a vertex labeled multi-set of functions $(F_{i,Y} : \mathbb{F}_2^n \to \mathbb{F}_2^n)_i$ and (c) a permutation $\pi \in S_n$. The function $F_{i,Y}$ updates the binary state of vertex i as a function of the states of vertex i and its Y-neighbors and leaves the states of all other vertices fixed. The permutation π represents a Y-vertex ordering according to which the functions $F_{i,Y}$ are applied. By composing the functions $F_{i,Y}$ in the order given by π we obtain the sequential dynamical system (SDS)

$$[\mathfrak{F}_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y} : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n.$$

Let $\mathbb{G}[\mathfrak{F}_Y, \pi]$ be the graph with vertex set $\mathbb{F}_2^n$ and edge set $\{(x, [\mathfrak{F}_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$. An SDS-morphism between $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Z, \sigma]$ is a triple (φ, η, Φ), where $\varphi : Y \longrightarrow Z$ is a graph-morphism, $\eta : S_{|Z|} \longrightarrow S_{|Y|}$ is a map such that $\eta(\sigma) = \pi$ and Φ is a digraph-morphism $\Phi : \mathbb{G}[\mathfrak{F}_Z, \sigma] \longrightarrow \mathbb{G}[\mathfrak{F}_Y, \pi]$. Our main result is that locally bijective graph-morphisms (coverings) between dependency graphs of SDS naturally induce SDS-morphisms. We show how these SDS-morphisms allow for a new proof for the upper bound on the number of inequivalent SDS obtained by only varying their underlying permutations. Here, two SDS are called inequivalent if they are inequivalent as dynamical systems. Furthermore, we apply our result in order to obtain phase space properties of SDS.
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1. Introduction and statement of results

Let Y be a loop-free, labeled, undirected graph with vertex set $\mathrm{v}[Y]=\{1,\ldots,n\}$ and edge set $\mathrm{e}[Y]$. We denote the set of Y-vertices adjacent to vertex i by $B_{0,Y}(i)$ and set $\delta_i=|B_{0,Y}(i)|$. The increasing sequence of elements of the sets $B_{0,Y}(i)$ and $B_{0,Y}(i)\cup\{i\}$ are referred to as

$$S_{1,Y}(i)=(j_1,\ldots,j_{\delta_i}),\quad B_{1,Y}(i)=(j_1,\ldots,i,\ldots,j_{\delta_i}), \tag{1}$$

and we set $d=\max_{1\leqslant i\leqslant n}\delta_i$. Each vertex i has an associated state $x_i\in\mathbb{F}_2$, and for each $k=1,\ldots,d+1$ we assume a symmetric function $f_{(k)}:\mathbb{F}_2^k\to\mathbb{F}_2$ to be given. The symmetry property will allow for more sequential dynamical system (SDS)-morphisms (see Definition 2) since the associated graph-morphisms preserve adjacencies but not the specific labeling of Y-neighborhoods. We denote the permutation group over k letters by S_k, set $\mathbb{N}_n=\{1,2,\ldots,n\}$ and use the sequence of Eq. (1) in order to introduce for $i\in\mathbb{N}_n$

$$\mathrm{proj}[i]:\mathbb{F}_2^n\to\mathbb{F}_2^{\delta_i+1},\quad (x_1,\ldots,x_n)\mapsto(x_{j_1},\ldots,x_i,\ldots,x_{j_{\delta_i}}). \tag{2}$$

For each Y-vertex i, we next define via $\mathrm{proj}[i]$ the *Y-local map* $F_{i,Y}$

$$y_i(x)=f_{(\delta_i+1)}\circ\mathrm{proj}[i](x), \tag{3}$$

$$F_{i,Y}(x)=(x_1,\ldots,x_{i-1},y_i(x),x_{i+1},\ldots,x_n). \tag{4}$$

By construction, $F_{i,Y}$ only changes the state of the ith coordinate of $(x_1,\ldots,x_i,\ldots,x_n)$ as a function of the states of i and all its Y-neighbors and we refer to the multi-set $(F_{i,Y})_i$ as $\mathfrak{F}_Y$. Clearly, for each $Y<K_n$ a given multi-set $(f_{(k)})_{1\leqslant k\leqslant n}$ induces a multi-set $\mathfrak{F}_Y$. We sometimes refer to the permutation π as the *update schedule* of the SDS.

Before we proceed with the definition of an SDS, let us present an example. Let Y be the circle graph over 4 vertices, i.e.,

$$Y=\mathrm{Circ}_4=\begin{array}{ccc}1 & \text{———} & 2\\ | & & |\\ 4 & \text{———} & 3.\end{array} \tag{5}$$

For each vertex $i=1,2,3,4$, we select the negation of the Boolean OR function $\mathrm{nor}_3:\mathbb{F}_2^3\to\mathbb{F}_2$ given by $\mathrm{nor}_3(0,0,0)=1$ and $\mathrm{nor}_3(x,y,z)=0$ for $(x,y,z)\neq(0,0,0)$. We introduce the Y-local maps

$$F_1(x)=(\mathrm{nor}(x_1,x_2,x_4),x_2,x_3,x_4),\ldots,$$

$$F_4(x)=(x_1,x_2,x_3,\mathrm{nor}(x_3,x_4,x_1))$$

and set $\pi=(1,2,3,4)$, according to which we update the vertices. For the initial state $(0,0,0,0)$ we compute

$$F_1(0,0,0,0)=(1,0,0,0),\quad F_2\circ F_1(0,0,0,0)=(1,0,0,0),\ldots,$$

$$F_4\circ F_3\circ F_2\circ F_1(0,0,0,0)=(1,0,1,0).$$

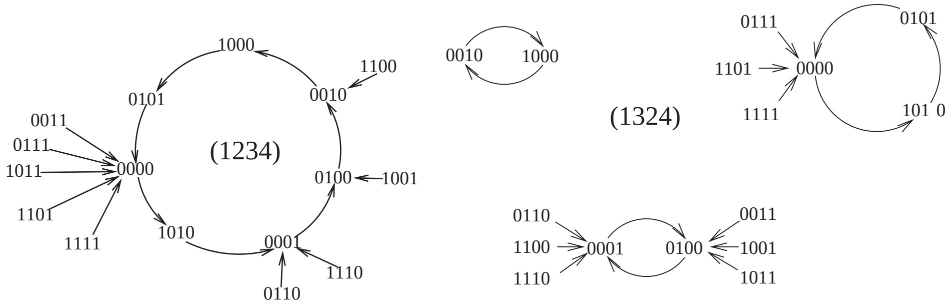

Fig. 1. The phase spaces of $[\mathbf{Nor}_{\mathrm{Circ}_4}, (1, 2, 3, 4)]$ (LHS) and $[\mathbf{Nor}_{\mathrm{Circ}_4}, (1, 3, 2, 4)]$ (RHS), respectively. Clearly, the phase spaces are not identical, and they are also nonisomorphic as directed graphs.

In Fig. 1 below we show all state-transitions obtained by iterating $F_4 \circ F_3 \circ F_2 \circ F_1$. We set $[\mathbf{Nor}_{\mathrm{Circ}_4}, (1, 2, 3, 4)] = F_4 \circ F_3 \circ F_2 \circ F_1$.

Definition 1. Let $[\mathfrak{F}_Y, \ ]$ be the mapping

$$[\mathfrak{F}_Y, \ ] : S_n \longrightarrow \mathbb{F}_2^{n\,\mathbb{F}_2^n}, \quad [\mathfrak{F}_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y}. \tag{6}$$

We call $[\mathfrak{F}_Y, \pi]$ the SDS over Y with respect to the ordering π.

Remark. In [1] certain generalizations regarding the update schemes are studied. Multiple updates of the local functions were allowed and it was shown how to retrieve the dependency graph from commutation relations among the local functions. In [6] a framework for SDS over *words* (i.e., update schedules with repetitions and omissions) is introduced. This combinatorial framework is based on certain equivalence classes of acyclic orientations of a generalized dependency graph which is induced by the underlying word w and Y. It turns out that this framework produces the theory of SDS over permutations as the generalized dependency graph is isomorphic to Y if the word is a permutation.

SDS can be analyzed from a purely combinatorial perspective by using the update graph $U(Y)$. $U(Y)$ has vertex set S_n and two vertices $(i_1, \ldots, i_n)$ and $(h_1, \ldots, h_n)$ are adjacent iff $(i_1, \ldots, i_n)$ and $(h_1, \ldots, h_n)$ differ by a transposition of two consecutive coordinates which are not connected by an edge in Y. Let $\sim_Y$ be the symmetric relation defined by

$$\pi \sim_Y \pi' \iff \pi, \pi' \text{ are connected by a } U(Y)\text{-path} \tag{7}$$

and set $[\pi]_Y = \{\pi' \mid \pi' \sim_Y \pi\}$. Obviously, we have

$$\forall \pi' \in [\pi]_Y; \qquad [\mathfrak{F}_Y, \pi] = [\mathfrak{F}_Y, \pi']. \tag{8}$$

Furthermore, an equivalence class $[\pi]_Y$ corresponds uniquely to an *acyclic orientation* $\mathfrak{O}$ of Y, i.e. a mapping $\mathfrak{O} : \mathrm{e}[Y] \longrightarrow \mathrm{v}[Y] \times \mathrm{v}[Y]$. We denote the set of all acyclic orientations of Y by $\mathsf{Acyc}(Y)$ and set $a(Y) = |\mathsf{Acyc}(Y)|$. The correspondence between equivalence classes of permutations and acyclic orientations

$$f(Y,\ ) : [S_n/\sim_Y] \longrightarrow \mathsf{Acyc}(Y), \tag{9}$$

has been proved in [4]. In fact, [6] proves the analog of Eq. (9) for SDS over arbitrary words. Taking the transitive closure, any $\mathfrak{O} \in \mathsf{Acyc}(Y)$ yields a unique multi-set of Y-independence sets $I_{\mathfrak{O}}(k)$, $k = 0, \ldots, m$. Since the elements of each $I_{\mathfrak{O}}(k)$ can be ordered linearly there exists the mapping $g(Y,\ ) : \mathsf{Acyc}(Y) \to S_n$ which assigns to an acyclic orientation its canonical permutation (see [4] for details). We then obtain the surjective mapping

$$h : \mathsf{Acyc}(Y) \longrightarrow [\mathfrak{F}_Y, S_n], \quad \mathfrak{O} \mapsto [\mathfrak{F}, g(Y, \mathfrak{O})]. \tag{10}$$

Let the graph Y and the multi-set $\mathfrak{F}_Y$ be fixed. Obviously, an SDS $[\mathfrak{F}_Y, \pi]$ induces a labeled digraph, $\mathbb{G}[\mathfrak{F}_Y, \pi]$, with vertex set $\mathbb{F}_2^n$ and edge set $\{(x, [\mathfrak{F}_Y, \pi](x)) | x \in \mathbb{F}_2^n\}$. We will call $\mathbb{G}[\mathfrak{F}_Y, \pi]$ the *phase space* of $[\mathfrak{F}_Y, \pi]$ and denote its set of vertices contained in cycles by $\mathsf{Per}[\mathfrak{F}_Y, \pi]$ and its vertices contained in cycles of length 1 by $\mathsf{Fix}[\mathfrak{F}_Y, \pi]$, respectively. Further, we call a $\mathbb{G}[\mathfrak{F}_Y, \pi]$-cycle a *periodic orbit* of $[\mathfrak{F}_Y, \pi]$.

One central question in SDS analysis and dynamical system analysis in general, is that of two SDS $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ being *equivalent*. Equivalence of SDS is defined with respect to a category $\mathfrak{C}[\mathfrak{F}_Y]$ whose objects are the digraphs $\mathbb{G}[\mathfrak{F}_Y, \pi]$.

For instance, let $\mathfrak{C}_{\mathsf{id}}[\mathfrak{F}_Y]$ be the category having only the identity as morphism. If h of Eq. (10) is bijective, which is for example the case if all local maps are induced by nor_k, then two SDS $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ are equivalent in $\mathfrak{C}_{\mathsf{id}}[\mathfrak{F}_Y]$, i.e., are equal as mappings, if and only if $[\pi]_Y = [\sigma]_Y$. In this paper, we will consider the category $\mathfrak{C}_{\mathrm{di}}[\mathfrak{F}_Y]$ having all digraph-morphisms as morphisms and therefore inducing the equivalence relation $\sim_{\mathrm{di}}$ given by $[\mathfrak{F}, \pi] \sim_{\mathrm{di}} [\mathfrak{F}, \pi']$ if and only if $\mathbb{G}[\mathfrak{F}, \pi] \cong \mathbb{G}[\mathfrak{F}, \pi']$. Fig. 1 shows the phase spaces of two SDS over $Y = \mathsf{Circ}_4$. We display the phase spaces of $[\mathbf{Nor}_{\mathsf{Circ}_4}, (1, 2, 3, 4)]$ and $[\mathbf{Nor}_{\mathsf{Circ}_4}, (1, 3, 2, 4)]$ and observe that just changing the underlying permutation can result in SDS having nonisomorphic phase spaces.

In [5,3] group actions on SDS are studied. These actions are induced by symmetries of Y and allow to investigate equivalence of SDS. We will show in Section 3 that Theorem 4 offers a new perspective and independent proof methods for the main results in [5,3].

2. Morphisms

It would be of fundamental interest to develop a “relative” theory of SDS, i.e., to be able to formulate a *category theory of* SDS. For this purpose, we have to define *morphisms* between SDS. Intuitively, a morphism concept should (a) incorporate the relation between the corresponding dependency graphs, (b) relate the corresponding permutations and (c) establish a relation between their phase space graphs. In [2] a fairly general concept of SDS-morphisms is outlined.

Definition 2. Let $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Z, \sigma]$ be two SDS. An SDS-morphism between $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Z, \sigma]$

$$(\varphi, \eta, \Phi) : [\mathfrak{F}_Y, \eta(\sigma)] \longrightarrow [\mathfrak{F}_Z, \sigma]$$

is a triple (φ, η, Φ), where $\varphi : Y \longrightarrow Z$ is a graph-morphism, $\eta : S_{|Z|} \longrightarrow S_{|Y|}$ is a mapping with the property $\eta(\sigma) = \pi$ and Φ is a digraph-morphism

$$\Phi : \mathbb{G}[\mathfrak{F}_Z, \sigma] \longrightarrow \mathbb{G}[\mathfrak{F}_Y, \pi].$$

If the maps φ, η and Φ are bijective, we call (φ, η, Φ) an SDS-isomorphism.

In general, graph-morphisms $m : Y \longrightarrow Z$ can map non-adjacent vertices into adjacent ones. This implies for a Z-local map $F_{m(i),Z}$ that it potentially depends on states of vertices which any local map $F_{i,Y}$ is independent of (as their associated vertices are not adjacent in Y to i). This observation motivates to consider *locally surjective graph-morphisms* (Eq. (12)). In case of local bijectivity both local maps $F_{m(i),Z}$ and $F_{i,Y}$ have exactly the same number of variables and we will show in Theorem 4 that there exists an SDS-morphism if $F_{m(i),Z}$ and $F_{i,Y}$ are equal as maps. In the following we will analyze locally surjective and locally bijective graph-morphisms, respectively. We will show in Lemma 3 that locally surjective graph-morphisms naturally induce the maps

$$\eta_\varphi : S_{|Z|} \longrightarrow S_{|Y|}.$$

An *acyclic orientation* $\mathfrak{O}$ of Y assigns to each Y-edge a direction such that the resulting directed graph $\mathfrak{O}(Y)$ is a tree. $\mathfrak{O}$ is a mapping

$$\mathfrak{O} : \mathrm{e}[Y] \longrightarrow \mathrm{v}[Y] \times \mathrm{v}[Y], \quad \mathfrak{O}(y) = (i(y), t(y)).$$

Let $\varphi : Y \longrightarrow Z$ be a graph-morphism. We set

$$\varphi(\mathfrak{O}_Y(y)) = (\varphi(i(Y)), \varphi(t(y)))$$

and introduce the set of *φ-symmetric acyclic orientations*

$$\begin{aligned}\mathsf{Acyc}^{\varphi}(Y) = \{\mathfrak{O} \in \mathsf{Acyc}(Y) \mid \forall z \in \mathrm{e}[Z];\\ \forall y, y' \in \varphi^{-1}(z);\ \varphi(\mathfrak{O}_Y(y)) = \varphi(\mathfrak{O}_Y(y'))\}.\end{aligned} \tag{11}$$

A graph-morphism $\varphi : Y \longrightarrow Z$ is called *locally surjective* and *locally bijective*, respectively, iff

$$\forall i \in \mathrm{v}[Y]; \quad \mathrm{res}(\varphi) : \mathrm{Star}_Y(i) \longrightarrow \mathrm{Star}_Z(\varphi(i)) \tag{12}$$

is surjective or bijective.

The following lemma will be instrumental for the proof of Theorem 4 as it relates the update schedules of two SDS. The key observation is that a locally surjective graph-morphism $\varphi : Y \longrightarrow Z$ induces a natural mapping from $\mathsf{Acyc}(Z)$ into $\mathsf{Acyc}(Y)$. For a graph X with $|X| = n$ we set

$$t_X : S_n \longrightarrow \mathsf{Acyc}(X);\ \pi \mapsto t_X(\pi) = f(X, [\pi]_X),$$

where $[\pi]_X$ denotes the $\sim_X$ equivalence class of π (Eq. (7)) and $f(X,\ )$ is defined in Eq. (9).

Lemma 3. *Let Y, Z be undirected, connected, loop-free graphs, $\varphi : Y \longrightarrow Z$ a locally surjective graph morphism and $|Z| = m$, $|Y| = n$. Then we have the one-to-one correspondence*

$$\psi_\varphi : \mathsf{Acyc}(Z) \longrightarrow \mathsf{Acyc}^\varphi(Y), \quad \mathfrak{O}_Z \mapsto \psi_\varphi(\mathfrak{O}_Z), \tag{13}$$

such that $\varphi(\psi_\varphi(\mathfrak{O}_Z)(y)) = \mathfrak{O}_Z(\varphi(y))$.

In particular, there is a natural embedding $\psi'_\varphi : \mathsf{Acyc}(Z) \longrightarrow \mathsf{Acyc}(Y)$, given by $\psi'_\varphi(\mathfrak{O}) = \psi_\varphi(\mathfrak{O})$.

Furthermore, let

$$\begin{aligned}\eta_\varphi &: S_m \to S_n, \quad \eta_\varphi(i_1, \ldots, i_m)\\ &= (j_1^{(i_1)}, \ldots, j^{(i_1)}_{|\varphi^{-1}(i_1)|}, \ldots, j_1^{(i_m)}, \ldots, j^{(i_m)}_{|\varphi^{-1}(i_m)|}),\end{aligned}$$

where $j_k^{(i)} \in \varphi^{-1}(i)$ and $j_k^{(i)} < j_{k+1}^{(i)}$. Then

$$\begin{tikzcd} S_m \arrow[r, "\eta_\varphi"] \arrow[d, "t_Z"'] & S_n \arrow[d, "t_Y"] \\ \mathsf{Acyc}(Z) \arrow[r, "\psi'_\varphi"] & \mathsf{Acyc}(Y) \end{tikzcd}$$

is a commutative diagram.

Proof. We first show that $\psi_\varphi : \mathsf{Acyc}(Z) \longrightarrow \mathsf{Acyc}^\varphi(Y)$ is well-defined. Suppose Y contains a cycle, C_Y, which is a directed cycle w.r.t. $\psi_\varphi(\mathfrak{O}_Z)$. Clearly, by restriction to the Z-subgraph $\varphi(C_Y)$, $\mathfrak{O}_Z$ induces the acyclic orientation $\mathfrak{O}'_Z$. Let ω_Z be an $\mathfrak{O}'_Z$-origin and $\omega_Y \in \varphi^{-1}(\omega_Z)$. Since $\omega_Y \in C_Y$, there exist two edges e_1, e_2 having ω_Y as terminus and origin and we have for $i = 1, 2$:

$$\varphi(\psi_\varphi(\mathfrak{O}_Z)(e_i)) = \mathfrak{O}_Z(\varphi(e_i))$$

which is impossible.

We proceed by proving that ψ_φ is bijective. We immediately conclude that ψ_φ is injective. To verify surjectivity we consider $\mathfrak{O}_Y \in \mathsf{Acyc}^\varphi(Y)$. Clearly, we obtain a ψ_φ-preimage of $\mathfrak{O}_Y$ by setting

$$\mathfrak{O}_Z(\{\varphi(i), \varphi(j)\}) = \varphi(\mathfrak{O}_Y(\{i, j\})).$$

It remains to show that $\mathfrak{O}_Z$ is acyclic. Again, suppose that Z contains a cycle C_Z which is a directed cycle w.r.t. $\mathfrak{O}_Z$. By restriction, $\varphi^{-1}(C_Z)$ induces the acyclic orientation $\mathfrak{O}'_Y$. Let ω_Y be an $\mathfrak{O}'_Y$-origin. Then there exist two Z-edges e_1, e_2 for which $\varphi(\omega_Y)$ is origin and terminus, respectively. Since φ is locally surjective, we can conclude that there are two Y-edges $\varepsilon_1, \varepsilon_2$ such that for $i = 1, 2$ we have $\mathfrak{O}_Z(e_i) = \varphi(\mathfrak{O}_Y(\varepsilon_i))$ which is, in view of ω_Y being an $\mathfrak{O}_Y$-origin, impossible.

The commutativity of the above diagram follows immediately and the proof of Lemma 3 is complete. $\square$

3. The main result

In this section, we will show that locally bijective and locally surjective graph-morphisms $\varphi : Y \longrightarrow Z$ induce SDS-morphisms. Before we state our main result, we introduce the Boolean functions $\mathrm{nor}_{(k)} : \mathbb{F}_2^k \to \mathbb{F}_2$ and $\mathrm{nand}_{(k)} : \mathbb{F}_2^k \to \mathbb{F}_2$ where

$$\mathrm{nor}_{(k)}(x_1, \ldots, x_k) = \begin{cases} 0 & \text{for } (x_1, \ldots, x_k) \neq (0, \ldots, 0), \\ 1 & \text{otherwise.} \end{cases}$$

and

$$\mathrm{nand}_{(k)}(x_1, \ldots, x_k) = \begin{cases} 1 & \text{for } (x_1, \ldots, x_k) \neq (1, \ldots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4. *Let Y, Z be connected loop-free graphs, $\varphi : Y \longrightarrow Z$ be a graph-morphism and let $\Phi : \mathbb{F}_2^{|Z|} \longrightarrow \mathbb{F}_2^{|Y|}$, where $\Phi(x)_k = x_{\varphi(k)}$. Then the following assertions hold*:

(a) *If $\varphi : Y \longrightarrow Z$ is locally bijective and $[\mathfrak{F}_Z, \pi]$ and $[\mathfrak{F}_Y, \eta_\varphi(\pi)]$ are induced by the set of local functions $f_{(k)} : \mathbb{F}_2^k \to \mathbb{F}_2$. Then $\Phi : \mathbb{G}[\mathfrak{F}_Z, \pi] \longrightarrow \mathbb{G}[\mathfrak{F}_Y, \eta_\varphi(\pi)]$ is a digraph-morphism and*

$$(\varphi, \eta_\varphi, \Phi) : [\mathfrak{F}_Y, \eta_\varphi(\pi)] \longrightarrow [\mathfrak{F}_Z, \pi] \tag{14}$$

is an SDS-morphism.

(b) *Let $\varphi : Y \longrightarrow Z$ be locally surjective and $[\mathfrak{F}_Z, \pi]$ and $[\mathfrak{F}_Y, \eta_\varphi(\pi)]$ be induced by $\mathrm{nor}_{(k)}$ or $\mathrm{nand}_{(k)}$. Then $\Phi : \mathbb{G}[\mathfrak{F}_Z, \pi] \longrightarrow \mathbb{G}[\mathfrak{F}_Y, \eta_\varphi(\pi)]$ is a digraph-morphism and*

$$(\varphi, \eta_\varphi, \Phi) : [\mathfrak{F}_Y, \eta_\varphi(\pi)] \longrightarrow [\mathfrak{F}_Z, \pi] \tag{15}$$

is an SDS-morphism.

Proof. We first prove assertion (a). Our goal is to show that $(\varphi, \eta_\varphi, \Phi) : [\mathfrak{F}_Y, \eta_\varphi(\pi)] \longrightarrow [\mathfrak{F}_Z, \pi]$ is a SDS-morphism. By assumption, $\varphi : Y \longrightarrow Z$ is its first component and according to Lemma 3, φ induces

$$\eta_\varphi : S_{|Z|} \longrightarrow S_{|Y|}$$

as the second component. In view of Definition 2 it remains to be proven that Φ is a digraph-morphism.

We will prove this in two steps. The first step is a purely local consideration and can be used to generalize Theorem 4 to words (i.e., update schedules with repetitions). In the second step, we verify with the help of Lemma 3 that our construction is compatible with the composition of local functions.

Claim 1. *Let $\varphi : Y \longrightarrow Z$ be a locally bijective graph morphism. Then we have the commutative diagram*

$$\begin{array}{ccc} \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\quad\Phi\quad} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \\ \Big\downarrow F_{\varphi(i),Z} & & \Big\downarrow \prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \\ \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\quad\Phi\quad} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \end{array} \tag{16}$$

$$\text{i.e.}\quad \Phi \circ F_{\varphi(i),Z} = \prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \circ \Phi.$$

Let us first analyze $\prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \circ \Phi$. $F_{i,Y}(\Phi(x))$ updates the state of i as a function of $((\Phi(x))_\ell, \ell \in B_{0,Y}(i) \cup \{i\})$. Since $(\Phi(x))_k = x_{\varphi(k)}$ we have

$$((\Phi(x))_\ell \mid \ell \in B_{0,Y}(i) \cup \{i\}) = (x_{\varphi(\ell)} \mid \ell \in B_{0,Y}(i) \cup \{i\}).$$

Local bijectivity implies that

$$\mathrm{res}(\varphi) : \mathrm{Star}_Y(i) \longrightarrow \mathrm{Star}_Z(\varphi(i))$$

is bijective and we obtain

$$(x_{\varphi(\ell)} \mid \ell \in B_{0,Y}(i) \cup \{i\}) = (x_{\varphi(\ell)} \mid \varphi(\ell) \in B_{0,Z}(\varphi(i)) \cup \{\varphi(i)\}).$$

Z is by assumption a loop-free graph, whence $\varphi^{-1}(\varphi(i))$ is an Y-independence set. Accordingly,

$$\prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y}$$

is a well-defined product of Y-local maps, without reference to some ordering, which updates all Y-vertices $j \in \varphi^{-1}(\varphi(i))$ based on $(x_{\varphi(\ell)} \mid \varphi(\ell) \in B_{0,Z}(\varphi(j)) \cup \{\varphi(j)\})$ to the state $(F_{i,Y}(\Phi(x)))_i$. Next, we compute $\Phi \circ F_{\varphi(i),Z}(x)$. By definition, $F_{\varphi(i),Z}(x)$ updates the state of the Z-vertex $\varphi(i)$ as a function of $(x_{\varphi(\ell)} \mid \varphi(\ell) \in B_{0,Z}(\varphi(i)) \cup \{\varphi(i)\})$. Further, we observe $(\Phi \circ F_{\varphi(i),Z}(x))_j = (F_{\varphi(i),Z}(x))_{\varphi(j)}$. That is, $\Phi \circ F_{\varphi(i),Z}(x)$ updates the states of the Y-vertices $j \in \varphi^{-1}(\varphi(i))$ to the state $(F_{\varphi(i),Z}(x))_{\varphi(i)}$. Since φ is locally bijective, we have for arbitrary Y-vertex i

$$(F_{i,Y}(\Phi(x)))_i = (F_{\varphi(i),Z}(x))_{\varphi(i)}$$

from which we conclude $\Phi \circ F_{\varphi(i),Z} = \prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \circ \Phi$.

Claim 2. *The diagram*

$$\begin{array}{ccc} \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\quad\Phi\quad} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \\ \Big\downarrow [\mathfrak{F}_Z,\pi] & & \Big\downarrow [\mathfrak{F}_Y,\eta_\varphi(\pi)] \\ \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\quad\Phi\quad} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \end{array}$$

is commutative.

Let $m = |\mathrm{v}[Z]|$ and $n = |\mathrm{v}[Y]|$. For $\pi = (i_1, \ldots, i_m)$ Lemma 3 implies

$$[\eta_\varphi(\pi)]_Y = [(\varphi^{-1}(i_1), \ldots, \varphi^{-1}(i_m))]_Y$$

and in view of Eq. (8) we obtain

$$[\mathfrak{F}_Y, \eta_\varphi(\pi)] = \prod_{j=1}^{m} \left[\prod_{d \in \varphi^{-1}(i_j)} F_{d,Y} \right].$$

We inductively apply $\prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \circ \Phi = \Phi \circ F_{\varphi(i),Z}$ and conclude

$$\prod_{j=1}^{m} \left[\prod_{d \in \varphi^{-1}(i_j)} F_{d,Y} \right] \circ \Phi = \Phi \circ \prod_{j=1}^{m} F_{i_j,Z},$$

whence Claim 2 and the proof of assertion (a) is complete.

Second, we prove assertion (b). We will restrict ourselves to the case of $[\mathfrak{F}_Z, \pi]$ and $[\mathfrak{F}_Y, \eta_\varphi(\pi)]$ being induced by $\mathrm{nor}_{(k)}$. The case of $\mathrm{nand}_{(k)}$ is proved analogously. As in the proof of (a) we know the first and second component of $(\varphi, \eta_\varphi, \Phi)$ by assumption and by Lemma 3, respectively. We proceed by showing that Φ is a digraph-morphism. For this purpose we set

$$U_{i,X}((x_1, \ldots, x_h)) = \{x_k \mid k \in (B_{0,X}(i) \cup \{i\}) \wedge x_k = 1\}.$$

Since $(f_k) = (\mathrm{nor}_{(k)})$, we immediately observe

$$\left(\prod_{d \in \varphi^{-1}(i_j)} F_{Y,d}(\Phi(x)) \right)_i = 1 \iff U_{i,Y}(\Phi(x)) = \emptyset, \tag{17}$$

$$(F_{Z,i_j}(x))_{i_j} = 1 \iff U_{i_j,Z}(x) = \emptyset. \tag{18}$$

Claim 3. *Let $\varphi : Y \longrightarrow Z$ be a locally surjective graph-morphism. Then we have the commutative diagram*

$$\begin{array}{ccc} \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\Phi} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \\ \Big\downarrow F_{\varphi(i),Z} & & \Big\downarrow \prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \\ \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\Phi} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \end{array} \quad \textit{i.e.}$$

$$\Phi \circ F_{\varphi(i),Z} = \prod_{j \in \varphi^{-1}(\varphi(i))} F_{j,Y} \circ \Phi.$$

To prove the claim we first show

$$U_{i,Y}(\Phi(x)) = \emptyset \iff U_{\varphi(i),Z}(x) = \emptyset. \tag{19}$$

Since $\varphi : Y \longrightarrow Z$ is a graph-morphism, we can immediately conclude

$$U_{i,Y}(\Phi(x)) \neq \emptyset \implies U_{\varphi(i),Z}(x) \neq \emptyset.$$

In order to prove

$$U_{\varphi(i),Z}(x) \neq \emptyset \implies U_{i,Y}(\Phi(x)) \neq \emptyset,$$

we need local surjectivity of $\varphi : Y \longrightarrow Z$. Let $U_{\varphi(i),Z}(x) \neq \emptyset$, i.e., there exists a vertex, k, with state $x_k = 1$ that is either adjacent to vertex $\varphi(i)$ in Z or $k = \varphi(i)$. Local surjectivity guarantees that there exists a vertex $k' \in \varphi^{-1}(k)$ adjacent to i in Y or $k' = i$ such that by definition of $\Phi : \mathbb{F}_2^{|Z|} \longrightarrow \mathbb{F}_2^{|Y|}$ we have $(\Phi(x))_{k'} = x_k = 1$, whence Eq. (19).

Since Φ lifts the state of the Z-vertex i_j to all Y-vertices contained in the independence set $\varphi^{-1}(i_j)$, we conclude from Eq. (19)

$$\Phi \circ F_{Z,\varphi(i)} = \prod_{j \in \varphi^{-1}(\varphi(i))} F_{Y,j} \circ \Phi,$$

whence Claim 3. □

We inductively apply $\Phi \circ F_{Z,\varphi(i)} = \prod_{j \in \varphi^{-1}(\varphi(i))} F_{Y,j} \circ \Phi$ and obtain with Lemma 3

$$\Phi \circ [\mathfrak{F}_Z, \pi] = \Phi \circ \prod_{j=1}^{m} F_{i_j,Z} = \prod_{j=1}^{m} \left[\prod_{d \in \varphi^{-1}(i_j)} F_{d,Y} \right] \circ \Phi = [\mathfrak{F}_Y, \eta_\varphi(\pi)] \circ \Phi$$

i.e., we have for locally surjective $\varphi : Y \longrightarrow Z$ the commutative diagram

$$\begin{array}{ccc} \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\Phi} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \\ \downarrow {\scriptstyle [\mathfrak{F}_Z,\pi]} & & \downarrow {\scriptstyle [\mathfrak{F}_Y,\eta_\varphi(\pi)]} \\ \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\Phi} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \end{array}$$

and assertion (b) follows.

Example. In order to illustrate Theorem 4 we consider two SDS with the local functions nor_3 and nor_4 over the graphs Y and Z displayed in Fig. 2(a). It is immediately clear that the mapping φ, which identifies the vertices x and x' for $x = a, b, \ldots, d$ is locally bijective. We obtain, according to Theorem 4, the two mappings η_φ and Φ. Fig. 2(b) illustrates the mapping η_φ and Fig. 2(c) shows how the unique component containing a 3-cycle of $\mathbb{G}[\mathfrak{F}_Z, (a, b, c, d)]$ embeds into $\mathbb{G}[\mathfrak{F}_Y, (a, a', b, b', c, c', d, d')]$.

In fact, $\mathbb{G}[\mathfrak{F}_Z, (a, b, c, d)]$ contains four 2-cycles and one 3-cycle while $\mathbb{G}[\mathfrak{F}_Y, (a, a', b, b', c, c', d, d')]$ has fourteen 2-cycles, one 3-cycle, two 4-cycles, two 6-cycles and eight 8-cycles.

Remark. In [6] SDS over words are studied. In such generalized update schedules it is possible that some local functions are updated multiple times. In the following we will consider a word as a multi-set $(a_1, \ldots, a_h)$, where each a_i is a Y-vertex. According to

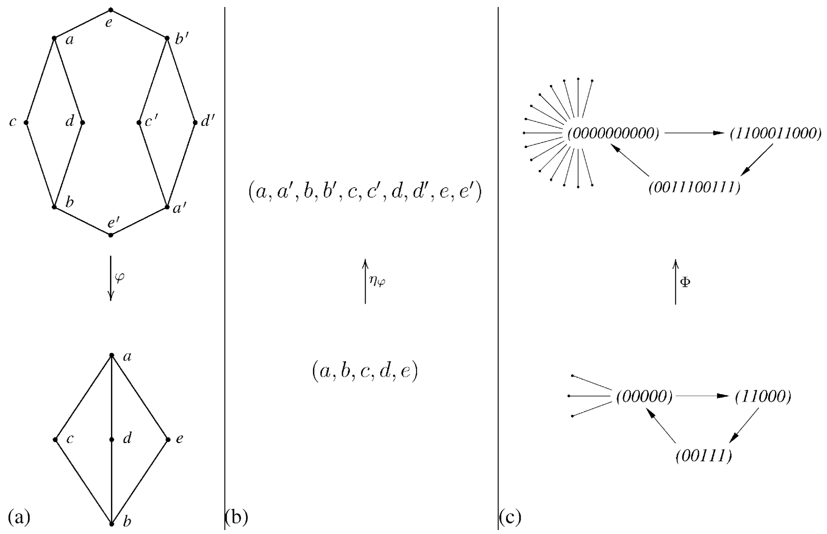

Fig. 2. As an illustration of Theorem 4 we present for given, locally bijective φ the mappings η_φ and Φ.

Claim 1 in the proof of Theorem 4 we have the commutative diagram

$$\begin{array}{ccc} \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\quad \Phi \quad} & \mathbb{F}_2^{|\mathrm{v}[Y]|} \\ \Big\downarrow F_{\varphi(i),Z} & & \Big\downarrow \prod_{j\in\varphi^{-1}(\varphi(i))} F_{j,Y} \\ \mathbb{F}_2^{|\mathrm{v}[Z]|} & \xrightarrow{\quad \Phi \quad} & \mathbb{F}_2^{|\mathrm{v}[Y]|}. \end{array}$$

We can use the above commutative diagram to generalize Theorem 4 to SDS over words as follows: Let $\varphi : Y \to Z$ be a locally bijective graph-morphism with $|\varphi^{-1}(1)| = k$ and $(a_1, \ldots, a_h)$ with $a_i \in \mathbb{N}_m$ be a multi-set. We set

$$\eta_\varphi(a_1, \ldots, a_h) = (\alpha_{a_1,1}, \ldots, \alpha_{a_1,k}, \ldots, \alpha_{a_h,1}, \ldots, \alpha_{a_h,k}),$$

where we have for any a_j, s: $\alpha_{a_j,s} \leqslant \alpha_{a_j,s+1}$ and $\{\alpha_{a_j,s} \mid 1 \leqslant s \leqslant k\} = \varphi^{-1}(a_j)$. Thus, it remains to verify that the analog of Claim 2 holds. A detailed proof of this generalization can be found in [6].

Theorem 4 immediately implies:

Corollary 5. *Let Y be a connected loop-free graph, φ a Y-automorphism, $\pi \in S_n$ and $[\mathfrak{F}_Y, \pi]$, $[\mathfrak{F}_Y, \eta_\varphi(\pi)]$ be* SDS *induced by the set of local functions $f_{(k)} : \mathbb{F}_2^k \to \mathbb{F}_2$. Then we have $\eta_\varphi(\pi) = \varphi^{-1} \circ \pi$ and we have an* SDS*-isomorphism*

$$(\varphi, \eta_\varphi, \Phi) : [\mathfrak{F}_Y, \pi] \longrightarrow [\mathfrak{F}_Y, \varphi \circ \pi], \tag{20}$$

where $\varphi \circ \pi = (\varphi(\pi(1)), \ldots, \varphi(\pi(n)))$.

In the following, we will show how we can use Theorem 4 and Corollary 5 to establish a group action of Y-automorphisms on SDS. We will then proceed by deriving an upper bound on the number of inequivalent SDS over a fixed graph Y and fixed multi-set of local functions $\mathfrak{F}_Y$.

Let $G = \mathsf{Aut}(Y)$ and $[\mathfrak{F}_Y, S_n] = \{[\mathfrak{F}_Y, \pi] \mid \pi \in S_n\}$. Then we have the mapping

$$\bullet : G \times [\mathfrak{F}_Y, S_n] \longrightarrow [\mathfrak{F}_Y, S_n], \tag{21}$$

$$(g, [\mathfrak{F}_Y, \sigma]) \mapsto g \bullet [\mathfrak{F}_Y, \sigma] = [\mathfrak{F}_Y, g \circ \sigma]. \tag{22}$$

To verify that this is an action we first observe that $g : Y \longrightarrow Y$ is as an Y-automorphism locally bijective. Secondly, we note that we have by construction

$$\eta_g(\pi) = g^{-1} \circ \pi = (g^{-1}(\pi(1)), \ldots, g^{-1}(\pi(n))).$$

Now we apply Theorem 4 and obtain the SDS-isomorphism

$$(\varphi, \eta_g, \Phi) : [\mathfrak{F}_Y, \pi] \longrightarrow [\mathfrak{F}_Y, g \circ \pi].$$

Hence, $[\mathfrak{F}_Y, g \circ \pi]$ is an SDS that is equivalent to $[\mathfrak{F}_Y, \pi]$ and in view of $\Phi(g)(x)_k = x_{g(k)}$ we have

$$g \circ [\mathfrak{F}_Y, \pi] \circ g^{-1} = [\mathfrak{F}_Y, g \circ \pi], \tag{23}$$

where $g(x_1, \ldots, x_n) = (x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)})$. We immediately conclude from Eq. (23) that $\bullet$ is a group action such that all elements of an G-orbit are equivalent SDS.

Next we observe that Y-automorphisms also act on acyclic orientations via

$$g\mathfrak{O}(\{i, k\}) = \mathfrak{O}(\{g^{-1}(i), g^{-1}(k)\}). \tag{24}$$

Furthermore,

$$h : \mathsf{Acyc}(Y) \longrightarrow [\mathfrak{F}_Y, S_n],$$

introduced in Eq. (10) is a G-map. Since $\bullet$ preserves equivalence classes of SDS it is well suited in order to compute an upper bound on the number of inequivalent SDS over a fixed graph Y and fixed multi-set of local maps $\mathfrak{F}_Y$. The key idea consists in applying Burnsides Lemma w.r.t. the G-action of Eq. (24). Burnsides Lemma relates N, the number of G-orbits with the cardinalities of the sets $\mathsf{Fix}(g)$ i.e., $N = (1/|G|)\sum_{g\in G} |\mathsf{Fix}(g)|$. In the following, we will show how Lemma 3 provides a combinatorial interpretation for the terms $\mathsf{Fix}(g)$.

Let G act on Y, i.e., G is a subgroup of $\mathsf{Aut}(Y)$. The projection into G-orbits,

$$\pi_G : Y \longrightarrow G\backslash Y, \quad i \mapsto G(i),$$

is a locally surjective graph-morphism. We set $\mathsf{Acyc}^G(Y) = \{\mathfrak{O} \in \mathsf{Acyc}(Y) \mid \forall g \in G;\ g\mathfrak{O} = \mathfrak{O}\}$ and observe that Lemma 3 implies that

$$\omega_G : \mathsf{Acyc}(G\backslash Y) \longrightarrow \mathsf{Acyc}^G(Y), \quad \overline{\mathfrak{O}} \mapsto \mathfrak{O}, \tag{25}$$

with $\pi_G(\mathfrak{O}(\{i,k\}))=\overline{\mathfrak{O}}(\{G(i),G(j)\})$ is a bijection. Setting $G=\langle g\rangle$, Eq. (25) provides us with the desired combinatorial interpretation of the terms $\mathsf{Fix}(g)=\mathsf{Acyc}^{\langle g\rangle}(Y)$ as the sets of acyclic orientations of the orbit-graphs of Y, $\langle g\rangle\backslash Y$, for $g\in G$.

Let Star$_n$ be the vertex-joint of the vertex 0 and the graph Circ$_n$. Further, let $|\mathbf{E}[Y,\mathfrak{F}_Y]|$ denote the number of inequivalent SDS for fixed base graph Y and multi-set of local functions $\mathfrak{F}_Y$. Burnsides Lemma and Eq. (25) imply

$$|\mathbf{E}[Y,\mathfrak{F}_Y]|\leqslant\frac{1}{|G|}\sum_{g\in G}|\mathsf{Fix}(g)|=\frac{1}{|G|}\sum_{g\in G}a(\langle g\rangle\backslash Y).$$

This bound is in fact sharp, since [3] proves

$$|\mathbf{E}[\mathsf{Star}_n,\mathbf{Nor}_{\mathsf{Star}_n}]| = \frac{1}{|\mathsf{Aut}(\mathsf{Star}_n)|}\sum_{\gamma\in\mathsf{Aut}(\mathsf{Star}_n)}|a(\langle\gamma\rangle\backslash\mathsf{Star}_n)|=n. \tag{26}$$

Accordingly, we have shown that Theorem 4 and its Corollary 5 imply

Theorem 6 (*Reidys [5]*). *Let Y be a connected loop-free graph and $\pi\in S_n$. Then we have*

$$|\mathbf{E}[Y,\mathfrak{F}_Y]|\leqslant\frac{1}{|\mathsf{Aut}(Y)|}\sum_{\gamma\in\mathsf{Aut}(Y)}|a(\langle\gamma\rangle\backslash Y)|, \tag{27}$$

$$|\mathbf{E}[\mathsf{Star}_n,\mathbf{Nor}_{\mathsf{Star}_n}]| = \frac{1}{|\mathsf{Aut}(\mathsf{Star}_n)|}\sum_{\gamma\in\mathsf{Aut}(\mathsf{Star}_n)}|a(\langle\gamma\rangle\backslash\mathsf{Star}_n)|=n. \tag{28}$$

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